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Disorder-induced temperature fluctuations in non-equilibrium systems

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Abstract. We study the effects on heat diffusion related to the presence of a quenched disorder of impurities at the boundaries. A simple lattice model summarizes the main characteristics of the surface through which the heat flux input is produced. The disorder induces temperature fluctuations which are proportional to the imposed temperature gradient and depend on the distribution of impurities. The existence of a non-linearity in the boundary conditions leads to the renormalization of the Nusselt number.

1. Introduction

Transport phenomena, or in particular diffusion in disordered systems, exhibit a great number of interesting effects that have been extensively studied during the last few years (for a review see, for example, [1]). The disorder is responsible for the appearance of anomalous diffusion as the diffusing particles may get trapped in some parts of the disordered geometry and hence the dynamic properties are modified. This is what happens, for example, in porous media [2] or in fractal structures [3]. Most of these subjects have been analysed with the use of discrete models on which a random walker is followed in time and the different probability densities are described by means of scaling laws.

Our purpose in this article is to address the problem of heat diffusion in systems in which the disorder enters the analysis through boundary conditions. We then assume that, in the process of heat diffusion, one of the boundaries of the system contains impurities which are randomly distributed in such a way that the heat transfer from the boundary to the bulk is a random quantity which induces temperature fluctuations in the system. We shall see that the correlation functions exhibit the peculiar behaviour of non-equilibrium fluctuations: they are proportional to the external gradients and they are long-ranged [4]. Moreover, the presence of non-linearities gives rise to the definition of a renormalized Nusselt number which depends on the random distribution of impurities at the boundary.

To this end we have organized the article as follows. In section 2 we describe our physical model. It consists of a system bounded by two parallel plates which are kept at different temperatures. One of the plates contains a random distribution of

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impurities which introduces a Gaussian stochastic process, delta-correlated in space. The underlying discrete model could be a lattice whose sites are occupied by two types of materials with different heat transfer coefficients which are randomly distributed in the lattice. In section 3 we show that the non-linearity in the fluctuations, introduced through boundary conditions at the plate, gives rise to a renormalized or effective Nusselt number. This dimensionless number controls the heat conduction in the bulk and contains corrections which are proportional to the intensity of the static noise. In section 4 we compute and analyse the stationary correlations and in the last section, we summarize our main results.

2. The model

We consider a system bounded by two plates at $y = 0, L$, and of infinite extent in the x and z directions. The upper boundary ($y = L$) is assumed to be kept at temperature T_L whereas the lower one ($y = 0$) is neither perfectly conducting nor perfectly insulating (see figure 1(a)). At this boundary the heat flux is specified through the Newton law of cooling [5]. Therefore, the stationary state can be identified from the solution of the boundary value problem defined through the differential equation [6]

$$\alpha \nabla^2 T_s(\boldsymbol{r}) = 0 \tag{2.1}$$

and the boundary conditions

$$\begin{aligned} &\lambda \left. \frac{\partial T_s(\boldsymbol{r})}{\partial y} \right|_{y=0} = \varepsilon(\boldsymbol{r}_\parallel)\{T_s(\boldsymbol{r}) - T_0)\}_{y=0} \qquad \text{(Neumann–Dirichlet)} \\ &T_s(\boldsymbol{r}) \,|_{y=L} = T_L \qquad \text{(Dirichlet).} \end{aligned} \tag{2.2}$$

Here $T_s(\boldsymbol{r})$ is the stationary temperature, α the thermal diffusivity, λ the thermal conductivity, $\varepsilon(\boldsymbol{r}_\parallel)$ the heat transfer coefficient, which is assumed to depend on the spatial coordinates parallel to the boundaries (x and z), and T_0 the temperature of the reservoir which is in contact with the boundary located at $y = 0$. This problem should be distinguished from the one treated in [7] in which we addressed the problem of thermal noise arising from fluctuating boundary conditions.

We will assume that the lower plate contains a random distribution of impurities in such a way that the heat transfer coefficient may be considered as the sum of the mean value ε_0 and a random contribution $\varepsilon_{\mathrm{R}}(\boldsymbol{r}_\parallel)$

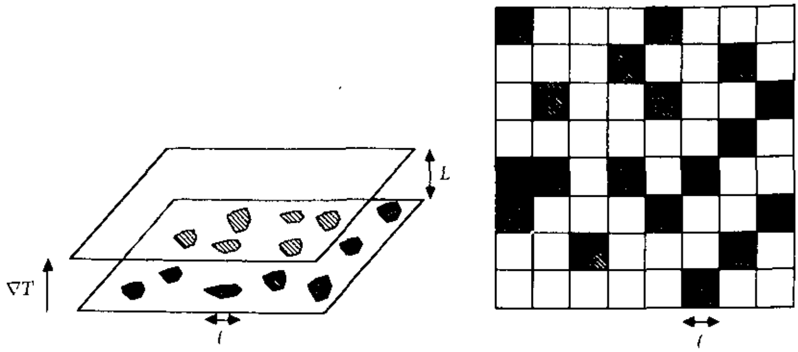
$$\varepsilon(\boldsymbol{r}_\parallel) = \varepsilon_0 + \varepsilon_{\mathrm{R}}(\boldsymbol{r}_\parallel). \tag{2.3}$$

The random term is modelled by a Gaussian stochastic process of zero mean and correlation

$$\langle \varepsilon_{\mathrm{R}}(\boldsymbol{r}_\parallel)\varepsilon_{\mathrm{R}}(\boldsymbol{r}'_\parallel)\rangle = A\delta(\boldsymbol{r}_\parallel - \boldsymbol{r}'_\parallel) \tag{2.4}$$

where A is a constant accounting for the intensity of the noise.

A very simple lattice model can be constructed keeping the essential features of our system (see figure 1(b)). One can imagine a lattice in which any site can be occupied with probability $p(1 - p)$ by some material of heat transfer coefficient ε_1



(a) (b)

Figure 1. (a) is a schematic drawing of our system. The space bounded by two parallel plates, in contact with heat sources at different temperatures, is filled by a heat conducting medium. In one of the boundaries, the quenched disorder of impurities is the origin of a stochastic heat flow from the plate to the system, giving rise to temperature fluctuations. In (b) we have represented the corresponding lattice model of the boundary. The sites of the lattice are occupied with probability $p(1-p)$ by some material of heat transfer coefficient $\varepsilon_1(\varepsilon_2)$. The distribution has a mean value ε_0.

(ε_2). Redefining ε_i, such that $\varepsilon_i \longrightarrow \varepsilon_i - \varepsilon_0$, it describes a stochastic process with zero mean and second moment

$$\langle \varepsilon(i)\varepsilon(j)\rangle = p(1-p)(\varepsilon_1 - \varepsilon_2)^2\delta_{ij} \tag{2.5}$$

where i and j denote lattice sites. This expression can be readily generalized to higher order moments where one notes that such a process is not Gaussian. The next step is to construct a coarse-grained version of this model in which Kronecker deltas become Dirac deltas multiplied by l^2, l being the lattice spacing or, in other words, the characteristic size of the impurities. In this case the non-Gaussian contributions become negligible and this model is suitably described by an intensity of the noise $A = p(1-p)(\varepsilon_1 - \varepsilon_2)^2 l^2$.

It is clear that the temperature at any point of the system will be affected by the presence of the impurities. To study this effect we must solve, first of all, the boundary value problem and obtain the formal solution for the temperature. To this end it is convenient to obtain the expression for the Green function [8], which is defined as the solution of

$$\alpha\left(\frac{\partial^2}{\partial y^2} - k_\parallel^2\right) G(y, y'; k_\parallel) = \delta(y - y') \tag{2.6}$$

which satisfies boundary conditions similar to (2.2), but now with ε_0 instead of ε, since ε_R enters the formal expression for the temperature,

$$\begin{aligned} &\lambda \frac{\partial}{\partial y} G(y, y'; k_\parallel)\bigg|_{y=0} = \varepsilon_0 G(y, y'; k_\parallel)_{y=0} \qquad \text{(Neumann–Dirichlet)} \\ &G(y, y'; k_\parallel)\,|_{y=L} = 0 \qquad \text{(Dirichlet).} \end{aligned} \tag{2.7}$$

The Green function satisfies the reciprocity relation consistent with (2.6)

$$G(y,y';k_\parallel)=G(y',y;k_\parallel). \tag{2.8}$$

The solution of (2.6)–(2.8) is found to be [9]

$$G(y,y';k_\parallel)=-\frac{1}{\alpha k_\parallel}\frac{\sinh[k_\parallel y_<]+(k_\parallel L/N)\cosh[k_\parallel y_<]}{\sinh[k_\parallel L]+(k_\parallel L/N)\cosh[k_\parallel L]}\sinh[k_\parallel(L-y_>)] \tag{2.9}$$

where $N\equiv\varepsilon_0 L/\lambda$ is the Nusselt number and use has been made of the definitions $y_>\equiv\max(y,y')$ and $y_<\equiv\min(y,y')$. Furthermore, we have introduced the Fourier transform in the parallel vector $r_\parallel$

$$\phi(r)=\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\mathrm{d}k_\parallel \mathrm{e}^{\mathrm{i}k_\parallel\cdot r_\parallel}\phi(k_\parallel,y) \tag{2.10}$$

where $k_\parallel=(k_x,k_z)$ and ϕ is an unspecified field.

Knowledge of the Green function enables one to arrive at the formal solution

$$T_s(k_\parallel,y)=(2\pi)^2\delta(k_\parallel)\theta_s(y)+\frac{\alpha}{\lambda}G(y,0;k_\parallel)\varepsilon_R(k_\parallel)\otimes\left[T_s(k_\parallel,0)-(2\pi)^2\delta(k_\parallel)T_0\right]. \tag{2.11}$$

This expression involves two contributions: the first one corresponds to the deterministic stationary profile according to (2.1) and (2.2), whereas the second, proportional to the random part ε_R, accounts for the presence of impurities at the boundary. In this last contribution $\varepsilon_R(k_\parallel)$ is a convolution operator whose action upon an unspecified function $\phi(k_\parallel)$ is defined as

$$\varepsilon_R(k_\parallel)\otimes\phi(k_\parallel)=\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\mathrm{d}k'_\parallel\varepsilon_R(k_\parallel-k'_\parallel)\phi(k'_\parallel). \tag{2.12}$$

Note that if the lower boundary is homogeneous ($\varepsilon_R=0$), equation (2.11) reduces to the stationary value

$$\theta_s(y)=\frac{N}{1+N}(T_L-T_0)\frac{y}{L}+\frac{T_L+NT_0}{1+N}. \tag{2.13}$$

This expression is similar to the stationary profile obtained when connecting the system with two thermal sources at constant temperatures. In our case, however, the temperature gradient is given by

$$|\nabla\theta_s|=\frac{N}{1+N}\nabla T \tag{2.14}$$

where $\nabla T\equiv(T_L-T_0)/L$ is the external gradient. Thus, the zero Nusselt limit (perfectly insulating boundary) corresponds to the equilibrium state, according to (2.13), in which the temperature is constant and equal to T_L. When N is very large (perfectly conducting boundary), equation (2.13) tends to the stationary temperature for a system under a temperature gradient.

3. Stationary mean value and renormalized Nusselt number

Our purpose in this section is to find an expression for the stationary mean value of the temperature. The starting point will be the formal solution (2.11). One should realize that its right-hand side depends on the temperature $T_s(\boldsymbol{k}_{\|}, 0)$ that follows from that equation by setting $y = 0$. Combining (2.11) with the expression for $T_s(\boldsymbol{k}_{\|}, 0)$, one arrives at

$$T_s(\boldsymbol{k}_{\|}, y) = (2\pi)^2\theta_s(y)\delta(\boldsymbol{k}_{\|}) + \frac{\alpha}{\lambda}G(0, y; \boldsymbol{k}_{\|})\varepsilon_{\rm R}(\boldsymbol{k}_{\|}) \otimes \frac{1}{1-(\alpha/\lambda)G(0,0;\boldsymbol{k}_{\|})\varepsilon_{\rm R}(\boldsymbol{k}_{\|})} \otimes (2\pi)^2\theta_s(0)\delta(\boldsymbol{k}_{\|}) \tag{3.1}$$

where we have taken $T_0 = 0$ without any loss of generality. Now we proceed to expand the propagator $\left[1-(\alpha/\lambda)G(0,0;\boldsymbol{k}_{\|})\varepsilon_{\rm R}(\boldsymbol{k}_{\|})\right]^{-1}$ in powers of $\varepsilon_{\rm R}$. We then obtain

$$T_s(\boldsymbol{k}_{\|}, y) = (2\pi)^2\theta_s(y)\delta(\boldsymbol{k}_{\|}) + \sum_{n=0}^{\infty}\left(\frac{\alpha}{\lambda}\right)^{n+1} G(0, y; \boldsymbol{k}_{\|})\varepsilon_{\rm R}(\boldsymbol{k}_{\|}) \underbrace{\otimes G(0,0;\boldsymbol{k}_{\|})\varepsilon_{\rm R}(\boldsymbol{k}_{\|}) \otimes \cdots \otimes G(0,0;\boldsymbol{k}_{\|})\varepsilon_{\rm R}(\boldsymbol{k}_{\|})}_{n \text{ times}} \otimes (2\pi)^2\theta_s(0)\delta(\boldsymbol{k}_{\|}). \tag{3.2}$$

Our next step is to take the average of (3.2). One gets

$$\begin{aligned}\langle T_s(\boldsymbol{k}_{\|}, y)\rangle = {} & (2\pi)^2\theta_s(y)\delta(\boldsymbol{k}_{\|}) \\ & + \sum_{n=0}^{\infty}\left(\frac{\alpha}{\lambda}\right)^{n+1} G(0,y;\boldsymbol{k}_{\|})\frac{1}{(2\pi)^{2(n+1)}}\int_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{k}_{\|1}\mathrm{d}\boldsymbol{k}_{\|2}\ldots\mathrm{d}\boldsymbol{k}_{\|n+1} \\ & \times \langle \varepsilon_{\rm R}(\boldsymbol{k}_{\|}-\boldsymbol{k}_{\|1})G(0,0;\boldsymbol{k}_{\|1})\varepsilon_{\rm R}(\boldsymbol{k}_{\|1}-\boldsymbol{k}_{\|2})\ldots G(0,0;\boldsymbol{k}_{\|n})\varepsilon_{\rm R}(\boldsymbol{k}_{\|n}-\boldsymbol{k}_{\|n+1})\rangle \\ & \times (2\pi)^2\theta_s(0)\delta(\boldsymbol{k}_{\|n+1}).\end{aligned} \tag{3.3}$$

Since $\varepsilon_{\rm R}$ is a Gaussian process with zero mean value, only the averages with an even number of terms will be different from zero. This fact leads to the expression

$$\begin{aligned}\langle T_s(\boldsymbol{k}_{\|}, y)\rangle = {} & (2\pi)^2\theta_s(y)\delta(\boldsymbol{k}_{\|}) \\ & + G(0,y;\boldsymbol{k}_{\|})\theta_s(0)\sum_{n=0}^{\infty}\left(\frac{\alpha}{\lambda}\right)^{2n+2}\frac{1}{(2\pi)^{2(2n+1)}}\int_{-\infty}^{+\infty}\mathrm{d}\boldsymbol{k}_{\|1}\mathrm{d}\boldsymbol{k}_{\|2}\ldots\mathrm{d}\boldsymbol{k}_{\|2n+2} \\ & \times G(0,0;\boldsymbol{k}_{\|1})\ldots G(0,0;\boldsymbol{k}_{\|2n+1}) \\ & \times \langle\varepsilon_{\rm R}(\boldsymbol{k}_{\|}-\boldsymbol{k}_{\|1})\varepsilon_{\rm R}(\boldsymbol{k}_{\|1}-\boldsymbol{k}_{\|2})\ldots\varepsilon_{\rm R}(\boldsymbol{k}_{\|2n+1}-\boldsymbol{k}_{\|2n+2})\rangle\delta(\boldsymbol{k}_{\|n+2}).\end{aligned} \tag{3.4}$$

Although the average involved in this equation may be evaluated for an arbitrary n, it is convenient, for the sake of clarity, to write the expression up to $n = 1$

$$\begin{aligned}\langle T_s(\boldsymbol{k}_{\|}, y)\rangle = {} & (2\pi)^2\theta_s(y)\delta(\boldsymbol{k}_{\|}) + \left(\frac{\alpha}{\lambda}\right)^2\theta_s(0)G(0,y;\boldsymbol{k}_{\|})\frac{1}{(2\pi)^2}\int\mathrm{d}\boldsymbol{k}_{\|1}\mathrm{d}\boldsymbol{k}_{\|2}G(0,0;\boldsymbol{k}_{\|1}) \\ & \times\langle\varepsilon_{\rm R}(\boldsymbol{k}_{\|}-\boldsymbol{k}_{\|1})\varepsilon_{\rm R}(\boldsymbol{k}_{\|1}-\boldsymbol{k}_{\|2})\rangle\delta(\boldsymbol{k}_{\|2}) \\ & + \left(\frac{\alpha}{\lambda}\right)^4\theta_s(0)G(0,y;\boldsymbol{k}_{\|})\frac{1}{(2\pi)^6}\int\mathrm{d}\boldsymbol{k}_{\|1}\mathrm{d}\boldsymbol{k}_{\|2}\mathrm{d}\boldsymbol{k}_{\|3}\mathrm{d}\boldsymbol{k}_{\|4} \\ & \times G(0,0;\boldsymbol{k}_{\|1})G(0,0;\boldsymbol{k}_{\|2})G(0,0;\boldsymbol{k}_{\|3}) \\ & \times\langle\varepsilon_{\rm R}(\boldsymbol{k}_{\|}-\boldsymbol{k}_{\|1})\varepsilon_{\rm R}(\boldsymbol{k}_{\|1}-\boldsymbol{k}_{\|2})\varepsilon_{\rm R}(\boldsymbol{k}_{\|2}-\boldsymbol{k}_{\|3})\varepsilon_{\rm R}(\boldsymbol{k}_{\|3}-\boldsymbol{k}_{\|4})\rangle\delta(\boldsymbol{k}_{\|4}).\end{aligned} \tag{3.5}$$

To proceed further with the development we will again use the fact that ε_R is Gaussian in order to factorize the fourth moment. One arrives at

$$
\begin{aligned}
\langle T_s(\boldsymbol{k}_\parallel, y)\rangle = (2\pi)^2\theta_s(y)\delta(\boldsymbol{k}_\parallel) + \left(\frac{\alpha}{\lambda}\right)^2 \theta_s(0)G(0,y;0)A\delta(\boldsymbol{k}_\parallel)\int \mathrm{d}\boldsymbol{k}_{\parallel 1} G(0,0;\boldsymbol{k}_{\parallel 1}) \\
+ \left(\frac{\alpha}{\lambda}\right)^4 \theta_s(0)G(0,y;0)\frac{1}{(2\pi)^2}A^2\delta(\boldsymbol{k}_\parallel)\int \mathrm{d}\boldsymbol{k}_{\parallel 1}\mathrm{d}\boldsymbol{k}_{\parallel 2} \\
\times \Big[G(0,0;0)G(0,0;\boldsymbol{k}_{\parallel 1})G(0,0;\boldsymbol{k}_{\parallel 2}) \\
+ G(0,0;\boldsymbol{k}_{\parallel 1})G(0,0;\boldsymbol{k}_{\parallel 2})G(0,0;\boldsymbol{k}_{\parallel 2}-\boldsymbol{k}_{\parallel 1}) \\
+ G(0,0,\boldsymbol{k}_{\parallel 1})G(0,0;\boldsymbol{k}_{\parallel 1})G(0,0;\boldsymbol{k}_{\parallel 2})\Big] .
\end{aligned}
\tag{3.6}
$$

This equation simplifies to

$$
\begin{aligned}
\langle T_s(\boldsymbol{k}_\parallel, y)\rangle = (2\pi)^2\theta_s(y)\delta(\boldsymbol{k}_\parallel) + \left(\frac{\alpha}{\lambda}\right)^2 \theta_s(0)G(0,y;0)(2\pi)^2AH_1\delta(\boldsymbol{k}_\parallel) \\
+ \left(\frac{\alpha}{\lambda}\right)^4 \theta_s(0)G(0,y;0)(2\pi)^2A^2\delta(\boldsymbol{k}_\parallel)\left[G(0,0;0)H_1^2 + H_2 + H_1H_3\right] + \cdots
\end{aligned}
\tag{3.7}
$$

where use has been made of the definitions

$$H_1 \equiv \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{k}_{\parallel 1} G(0,0;\boldsymbol{k}_{\parallel 1}) \tag{3.8}$$

$$H_2 \equiv \frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{k}_{\parallel 1}\mathrm{d}\boldsymbol{k}_{\parallel 2} G(0,0;\boldsymbol{k}_{\parallel 1})G(0,0;\boldsymbol{k}_{\parallel 2})G(0,0;\boldsymbol{k}_{\parallel 2}-\boldsymbol{k}_{\parallel 1}) \tag{3.9}$$

$$H_3 \equiv \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{k}_{\parallel 1} \left[G(0,0;\boldsymbol{k}_{\parallel 1})\right]^2 . \tag{3.10}$$

From the definition of the propagator (cf equation (2.9)) one concludes that the integrals introduced through (3.8)–(3.10) diverge for large wavenumbers; for this reason we must introduce a cutoff wavenumber k_c which is the inverse of a characteristic microscopic length of the interface that may be identified with the mean size of the impurities (l). Thus the integrals behave as $H_1 \sim k_cL$, $H_2 \sim k_cL\ln(k_cL)$ and $H_3 \sim \ln(k_cL)$, for $N < k_cL$. Taking this fact into account, it turns out that the main contribution in the term proportional to A^2 is H_1^2, the same holding for successive terms in the expansion. This consideration enables us to recover the general expression for the temperature

$$\langle T_s(\boldsymbol{k}_\parallel, y)\rangle = (2\pi)^2\delta(\boldsymbol{k}_\parallel)\left\{\theta_s(y) + \frac{G(0,y;0)}{G(0,0;0)}\theta_s(0)\sum_{n=1}^{\infty}\left[\left(\frac{\alpha}{\lambda}\right)^2 AH_1G(0,0;0)\right]^n\right\} . \tag{3.11}$$

There exists a physical reason for we restricting ourselves to the case $N \ll k_cL$. From the definition of the Nusselt number and the fact that $k_c \sim l^{-1}$, one arrives at

$N/k_c L = l/(\lambda/\varepsilon_0)$ where λ/ε_0 is a characteristic length parallel to the boundary. In order for the coarse-graining procedure to make physical sense it is necessary for the ratio $l/(\lambda/\varepsilon_0)$ to be much less than one.

Substituting $\theta_s(y)$ and the propagators for their explicit values we get

$$\langle T_s(\boldsymbol{k}_\parallel, y)\rangle = (2\pi)^2\delta(\boldsymbol{k}_\parallel)\frac{T_L}{1+N}\left[\left\{1+\sum_{n=1}^{\infty}\left[\frac{-\alpha L}{\lambda^2}\frac{AH_1}{1+N}\right]^n\right\}\right.$$
$$\left. + \left\{N - \sum_{n=1}^{\infty}\left[\frac{-\alpha L}{\lambda^2}\frac{AH_1}{1+N}\right]^n\right\}\frac{y}{L}\right] \tag{3.12}$$

which, taking the series as an expansion in powers of the term within the innermost square bracket, may be written in a much simpler form as follows

$$\langle T_s(\boldsymbol{k}_\parallel, y)\rangle = (2\pi)^2\delta(\boldsymbol{k}_\parallel)\frac{T_L}{1+N+\frac{\alpha L}{\lambda^2}AH_1}\left\{1+\left(N+\frac{\alpha L}{\lambda^2}AH_1\right)\frac{y}{L}\right\}. \tag{3.13}$$

This equation is identical to the stationary and noiseless temperature (2.13) with a renormalized Nusselt number

$$N^R \equiv N - \frac{AJ}{\lambda^2} \tag{3.14}$$

where we have introduced J as a dimensionless integral proportional to H_1

$$J = \frac{1}{2\pi}\int_0^{k_c L}\frac{z\sinh z}{N\sinh z + z\cosh z}\mathrm{d}z \tag{3.15}$$

which may be approximated by $k_c L$, for $N < k_c L$. In this case, we finally obtain

$$N^{\mathrm{R}} = N\left(1 - \frac{A}{\lambda^2}\frac{k_c L}{N}\right). \tag{3.16}$$

This effective Nusselt number depends on the physical properties of the bulk, such as heat conductivity and length, as well as on those due to the presence of the impurities at the boundaries by means of the intensity of the noise and the cutoff wavenumber. When replacing A by its value in terms of the microscopic interface properties, one gets

$$N^{\mathrm{R}} = N\left(1 - p(1-p)\frac{(\varepsilon_1-\varepsilon_2)^2}{\varepsilon_0^2}\frac{N}{k_c L}\right) \tag{3.17}$$

for which the largest correction is obtained when $p = 0.5$, $\varepsilon_2 = 0$, and therefore $\varepsilon_0 = \varepsilon_1/2$. For $N/k_c L \simeq 0.1$ this correction is about 10%. As we shall see in the next section, the effect of the impurities is important only in a layer close to the surface. Thus, since N accounts for an overall effect, it is reasonable to get a small correction to the Nusselt number.

4. Static correlation function

In this section we will compute the static correlation function from the expression for the stationary temperature (2.11), in which we keep $T_0 \neq 0$ in order to make a comparison with thermal noise. We will perform such a calculation up to the lowest order in the intensity of the noise; in this case, different contributions arise and the correlation function reads

$$\begin{aligned}\langle T_s(\boldsymbol{k}_\parallel, y) T_s(\boldsymbol{k}'_\parallel, y')\rangle &= (2\pi)^4 \theta_s(y)\theta_s(y')\delta(\boldsymbol{k}_\parallel)\delta(\boldsymbol{k}'_\parallel) \\ &+ (2\pi)^4 \left(\frac{\alpha}{\lambda}\right)^2 A H_1 \left[\theta_s(0) - T_0\right] \left[\theta_s(y) G(0, y'; 0)\right. \\ &\left. + \theta_s(y') G(0, y; 0)\right] \delta(\boldsymbol{k}_\parallel)\delta(\boldsymbol{k}'_\parallel) \\ &+ (2\pi)^2 \left(\frac{\alpha}{\lambda}\right)^2 A \left[\theta_s(0) - T_0\right]^2 G(0, y'; k_\parallel) G(0, y; k_\parallel) \delta(\boldsymbol{k}_\parallel + \boldsymbol{k}'_\parallel)\end{aligned} \qquad (4.1)$$

where the integral H_1 has been defined in the previous section. Transforming back to the real space, we get

$$\begin{aligned}\langle T_s(\boldsymbol{r}) T_s(\boldsymbol{r}')\rangle &= \theta_s(y)\theta_s(y') \\ &- \frac{2\alpha A H_1}{\lambda^2} \left(\frac{T_L - T_0}{1+N}\right)^2 \frac{L}{1+N} \left[1 - N\frac{yy'}{L^2} + \frac{(N-1)(y+y')}{2L}\right] \\ &+ \left(\frac{\alpha}{\lambda}\right)^2 \left(\frac{T_L - T_0}{1+N}\right)^2 A \frac{1}{(2\pi)^2} \int \mathrm{d}\boldsymbol{k}_\parallel \mathrm{e}^{\mathrm{i}\boldsymbol{k}_\parallel \cdot (\boldsymbol{r}_\parallel - \boldsymbol{r}'_\parallel)} G(0, y; k_\parallel) G(0, y'; k_\parallel).\end{aligned} \qquad (4.2)$$

Note that in this expression, the correlation function is translational invariant only in a plane parallel to the boundaries. After some straightforward manipulations, one may arrive at the static temperature correlation function

$$\begin{aligned}&\langle \delta T(\boldsymbol{r}) \delta T(\boldsymbol{r}')\rangle \\ &\quad = \left(\frac{\alpha}{\lambda}\right)^2 \left(\frac{T_L - T_0}{1+N}\right)^2 A \frac{1}{(2\pi)} \int_0^\infty k_\parallel \mathrm{d}k_\parallel J_0(k_\parallel |\boldsymbol{r}_\parallel - \boldsymbol{r}'_\parallel|) \\ &\quad \times G(0, y; k_\parallel) G(0, y'; k_\parallel)\end{aligned} \qquad (4.3)$$

where $\delta T(\boldsymbol{r}) \equiv T_s(\boldsymbol{r}) - \theta_s(y)$ is the temperature fluctuation and J_0 is the Bessel function of first kind and zeroth order. From this last expression, one may arrive at

$$\langle \delta T(\boldsymbol{r}) \delta T(\boldsymbol{r}')\rangle = \left(\frac{1}{\varepsilon_0}\right)^2 (\nabla T)^2 A I(y, y'; \tau; N) \qquad (4.4)$$

where use has been made of the definition

$$I(y, y'; \tau; N) \equiv \alpha^2 \left(\frac{N}{1+N}\right)^2 \int_0^{k_c L} k_\parallel \mathrm{d}k_\parallel J_0(k_\parallel \tau) G(0, y; k_\parallel) G(0, y'; k_\parallel). \qquad (4.5)$$

Here $\tau \equiv |\boldsymbol{r}_\parallel - \boldsymbol{r}'_\parallel|$ and we have introduced the cutoff wavevector, related to the discrete nature of the interface, as in the previous section. Instead of solving for I in a general situation, we will illustrate the behaviour of the integral in two interesting and representative cases.

4.1. Correlations at the surface

In this case we will consider $y = y' = 0$ and $\tau \neq 0$. If $\tau/L \gg 1$, then $J_0(z\tau/L) \ll 1$, with $z \in [0, k_c L]$, and therefore $I_1(\tau, N) \equiv I(0, 0; \tau; N) \ll 1$. On the other hand when $\tau/L \gg 1$ the integral I_1 is found to behave as

$$I_1(\tau/L \ll 1, N) \simeq \left(\frac{N}{N+1}\right)^2 [A(N)\log_{10}(\tau/L) + B(N)] \tag{4.6}$$

where A and B are unknown functions of the Nusselt number. In figure 2 we have represented I_1 as a function of the decimal logarithm of τ/L, for different Nusselt numbers. According to the plot, we realize that $A(N) \simeq -2$. These results show that the correlations at the surface decrease smoothly with the ratio τ/L and have important values for $N \simeq 1 - 10^2$. This fact must be emphasized because it points out that the non-equilibrium state of the system makes the delta-correlated noise at the surface to transform into a long-ranged noise in the bulk.

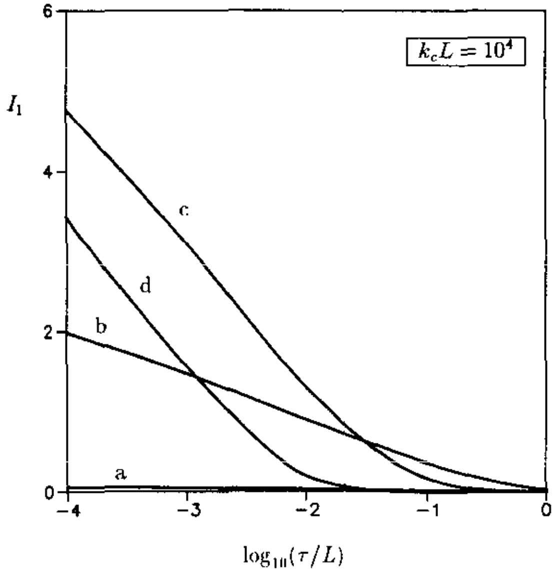


Figure 2. Long-ranged correlations at the surface. The integral I_1 is plotted as a function of the decimal logarithm of the ratio τ/L, for different Nusselt numbers. Curves labelled (a), (b), (c) and (d) correspond to $N = 0.1$, 1, 10 and 100, respectively.

Finally, from (4.5) we conclude that these correlations are practically insensitive to the particular value of $k_c L$. In fact, one has

$$\frac{\partial I_1}{\partial k_c L} \simeq \frac{1}{k_c L}\left(\frac{N}{1+N}\right)^2 \ll 1 \tag{4.7}$$

provided that the condition $k_c L \gg 1$ holds.

4.2. Correlations in the bulk

Here we will analyse the correlations for $y = y'$ and $\tau = 0$. In this case, the integral I becomes

$$I_2(y,N) \equiv I(y,y'=y;0;N) = \left(\frac{N}{N+1}\right)^2 \int_0^{k_c L} \mathrm{d}z\, z \left\{\frac{\sinh[z(1-\bar{y})]}{N\sinh z + z\cosh z}\right\}^2 \qquad (4.8)$$

where $\bar{y} \equiv y/L$. To study the behaviour of I_2 one should first realize that

$$\frac{\partial I_2}{\partial k_c L} \simeq \frac{1}{k_c L}\left(\frac{N}{1+N}\right)^2 \ll 1 \qquad (4.9)$$

just as in the previous case; moreover, it is easy to show that $I_2(\bar{y}=1,N)=0$, $\partial I_2/\partial\bar{y}|_{\bar{y}\neq 1} < 0$ and $\partial I_2/\partial\bar{y}|_{\bar{y}=1} = 0$. To analyse the dependence of I_2 with $\bar{y}$ it is convenient to separate the integral appearing in (4.8) from the prefactor containing the Nusselt number. Accordingly, we define $\tilde{I}_2 \equiv I_2(N+1)^2/N^2$. Then, one may also show from (4.8) that $\tilde{I}_2(\bar{y}=0,N=0) \simeq \log(k_c L)$, and

$$\left.\frac{\partial \tilde{I}_2}{\partial \bar{y}}\right|_{\bar{y}=0;N=0} = -2\log[\cosh(k_c L)] \simeq -2k_c L \qquad (4.10)$$

$$\left.\frac{\partial \tilde{I}_2}{\partial \bar{y}}\right|_{\bar{y}=1/2;N=0} = \frac{1}{\cosh(k_c L)} - 1 \simeq -1. \qquad (4.11)$$

Equation (4.10) indicates that the decay of the correlations with the ratio y/L is very fast in the neighbourhood of $y=0$, but this is no longer true away from this point, as follows from (4.11). Although $N=0$ is not a representative case, since $I_2 \equiv 0$, it is the only value from which analytical results can be obtained and the behaviour for larger Nusselt numbers is not qualitatively modified. In figure 3 we have represented I_2 as a function of $\bar{y}$, for different Nusselt numbers. In view of this plot, we conclude that *the noise is important in a macroscopic layer around* $y=0$, *the thickness of which*, δ, being a function of the Nusselt number. Due to the prefactor in (4.8) containing the Nusselt number, δ amounts to negligible values for $N<1$, but takes important values in the opposite case, the maximum taking place for $N \simeq 1$. To investigate this dependence we will compute the average of the integral $\tilde{I}_2$ over the whole system

$$\langle \tilde{I}_2(N)\rangle \equiv \frac{1}{L}\int_0^L \mathrm{d}y\, \tilde{I}_2(y,N). \qquad (4.12)$$

The double integral involved in this expression can be analytically computed, arriving at

$$\langle \tilde{I}_2(N)\rangle = \frac{1}{2}\left[\frac{1}{N+1} - \frac{\tanh(k_c L)}{N\tanh(k_c L) + k_c L}\right] \simeq \frac{1}{2(N+1)}. \qquad (4.13)$$

This expression indicates that δ depends smoothly on N for $N \ll 1$, and decreases as N^{-1} in the opposite case $N > 1$. Plots in figure 3 show that the proper behaviour of

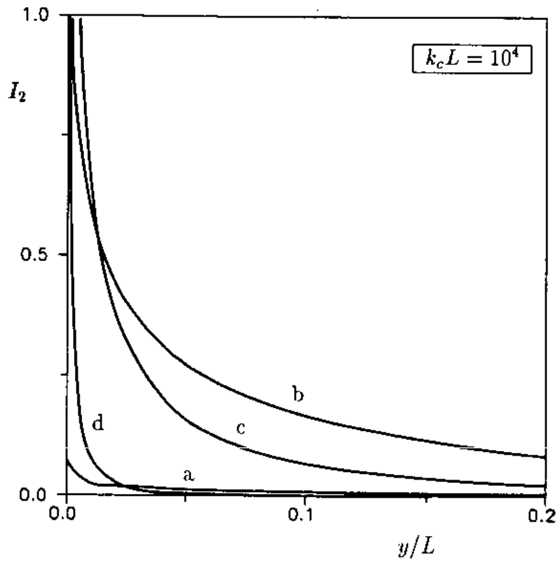


Figure 3. The integral I_2 accounting for the correlations in the bulk is plotted as a function of the dimensionless coordinate $\tilde{y}$, for different Nusselt numbers. Labels (a), (b), (c) and (d) indicate $N = 0.1$, 1, 10 and 100, respectively. When $\tilde{y}$ approaches 0, I_2 grows very rapidly, but $I_2(\tilde{y} = 0, N)$ remains finite. It vanishes for $N = 0$ whereas for the other values we have chosen $I_2(\tilde{y} = 0, N = 0.1) \simeq 0.1$, $I_2(\tilde{y} = 0, N = 1) \simeq 2.0$, $I_2(\tilde{y} = 0, N = 10) \simeq 4.9$, and $I_2(\tilde{y} = 0, N = 100) \simeq 3.5$.

I_2 is a balance between this N-dependence of δ and the fast decay for $N < 1$ due to the prefactor $N^2/(N+1)^2$ appearing in (4.8) in addition of $\tilde{I}_2$.

Expression (4.13) also allows us to illustrate the importance of the disorder-induced noise by comparing the average of (4.4) over the whole system with the mean-square fluctuation of thermal noise [10]: $\langle(\delta T)^2\rangle_0 = k_B \alpha T_0^2/\lambda L$, k_B being the Boltzmann constant. We then get

$$\frac{\langle(\delta T)^2\rangle}{\langle(\delta T)^2\rangle_0} = p(1-p)\left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_0}\right)^2 \left(\frac{\beta}{k_c L}\right)^2 \frac{L\lambda}{k_B \alpha} \frac{N^2}{2(N+1)^3} \tag{4.14}$$

where we have introduced the non-equilibrium parameter $\beta = (T_L - T_0)/T_0$. When $p = 0.5$, $\epsilon_2 = 0$ and $\epsilon_0 = \epsilon_1/2$, and for typical values of the other involved parameters, $\lambda/\alpha \simeq 10^{-5}$ J m^{-1} K^{-1}, $L \simeq 10^{-2}$ m, $\beta \simeq 10^{-1}$ and $k_c L \simeq 10^4$, this ratio yields

$$\frac{\langle(\delta T)^2\rangle}{\langle(\delta T)^2\rangle_0} \simeq 10^6 \frac{N^2}{(N+1)^3} \tag{4.15}$$

which clearly indicates the predominance of quenched noise over the thermal one, specially in the regime $N \simeq 1$.

5. Discussion

In this article we have analysed the phenomenon of disorder-induced temperature fluctuations in non-equilibrium systems. To this purpose we have introduced a model accounting for the process of heat transfer in a system between two parallel plates. One of them is modelled as a lattice whose sites are occupied, with different probabilities, by grains of two materials with different heat transfer coefficients. The random occupancy then gives rise to a stochastic process for the heat transfer coefficient of the plate whose correlation is given by equation (2.4). This correlation exhibits a maximum value when the probabilities are equal and for very different values of the heat transfer coefficients, provided that the mean heat transfer coefficient is fixed.

The stochastic process defined at the plate induces temperature fluctuations whose correlations are proportional to $(\nabla T)^2$ and $(\varepsilon_1 - \varepsilon_2)^2$. We then conclude that this effect takes place in non-equilibrium systems under a temperature gradient and disappears when both materials have the same heat transfer coefficient. Moreover, these correlations are more important than the ones originating from thermal noise and exhibit long-range behaviour [4].

Another interesting aspect we have analysed here is the renormalization of the Nusselt number due to a non-linearity whose origin is the presence of quenched fluctuations at the interface. This dimensionless number, controlling the heat transfer, is then modified by a term which is proportional to the intensity of the noise.

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